

**PARABOLIC GEOMETRIES  
FOR PEOPLE THAT LIKE PICTURES**

**LECTURE 6: INTERPRETING THE KILLING FORM  
(PART 2: NONCOMPACT SYMMETRIC SPACES)**

JACOB W. ERICKSON

Whenever  $G$  is a Lie group with finitely many connected components, a fundamental result (see, for example, Theorem 14.1.3 of [3]) in the general structure theory of Lie groups tells us that  $G$  has a maximal compact subgroup, and that all maximal compact subgroups are conjugate to each other. Moreover, maximal compact subgroups contain all of the nontrivial aspects of the topology of  $G$ : for  $K \leq G$  a maximal compact subgroup,  $G$  is diffeomorphic—though generally not isomorphic—to the product of  $K$  with a vector space.

We saw this, for example, in Euclidean geometry:  $I(2)$ , viewed as the orthonormal frame bundle over  $\mathbb{R}^2$ , was clearly diffeomorphic to  $\mathbb{R}^2 \times O(2)$ ; indeed, it was isomorphic to  $\mathbb{R}^2 \rtimes O(2)$ . Here,  $O(2)$  is a maximal compact subgroup of  $I(2)$ , and all other maximal compact subgroups of  $I(2)$  are conjugate to  $O(2)$ .

As we said earlier, if we want to find geometrically interesting models, then it makes sense to look for Lie-theoretically interesting models. Choosing our isotropy to be a maximal compact subgroup  $K$  is amongst the most Lie-theoretically interesting choices we can make in this case, and if, moreover, we choose our model group  $G$  to be a semisimple Lie group, then the underlying geometry is, as we would expect, remarkably deep. Such models  $(G, K)$  correspond to Riemannian symmetric spaces of noncompact type, and the Killing form gives several key tools for studying them, including:

- A  $\mathfrak{h}$ -orthogonal decomposition of the Lie algebra of the model group, called the *Cartan decomposition*
- A canonical Riemannian metric and notion of distance on  $G/K$
- A convenient description of the stabilizers of “points at infinity”

While this doesn’t directly help us understand the Killing form unless we already have experience with symmetric spaces, it does let us visualize several important interactions between the Killing form and the underlying representation theory. In particular, it will give us insight into what parabolic subgroups look like, and in the next lecture, we will explore our algebraic definition of parabolic subgroups and connect it to the more immediately geometric idea of “points at infinity”.

## 1. RIEMANNIAN SYMMETRIC SPACES OF NONCOMPACT TYPE

For the rest of the lecture, let us fix a model  $(G, K)$ , where  $G$  is a semisimple Lie group with finitely many connected components such that the identity component  $G^\circ$  has finite center and  $K$  is a maximal compact subgroup. In this case, the notion of *Killing perpendiculars* gives us a very convenient description of the topological decomposition of  $G$  as a product of  $K$  with a vector space.

**Definition 1.1.** For a subspace  $V \subseteq \mathfrak{g}$ , its *Killing perpendicular* (or *Killing perp*) is the subspace

$$V^\perp := \{X \in \mathfrak{g} : \mathfrak{b}(X, v) = 0 \text{ for each } v \in V\}.$$

In our case, the Lie algebra  $\mathfrak{g}$  decomposes, as a vector space, as a  $\mathfrak{b}$ -orthogonal direct sum  $\mathfrak{k}^\perp \oplus \mathfrak{k}$ , where the subspace  $\mathfrak{k}^\perp$  is the Killing perp of the Lie subalgebra  $\mathfrak{k}$  corresponding to  $K$ . The exponential map restricts to an *embedding* on  $\mathfrak{k}^\perp$ , so that  $\exp(\mathfrak{k}^\perp)$  is diffeomorphic to  $\mathfrak{k}^\perp$ , and moreover, the map  $\mu : \exp(\mathfrak{k}^\perp) \times K \rightarrow G$  given by applying the group operation  $(\exp(X), k) \mapsto \exp(X)k$  is a diffeomorphism. In particular, the usual quotient map  $q_K : G \rightarrow G/K$  restricts to a diffeomorphism from  $\exp(\mathfrak{k}^\perp)$  to  $G/K$ , and we get a projection map

$$\text{pr} : G \rightarrow \exp(\mathfrak{k}^\perp),$$

which induces a section of  $q_K$ .

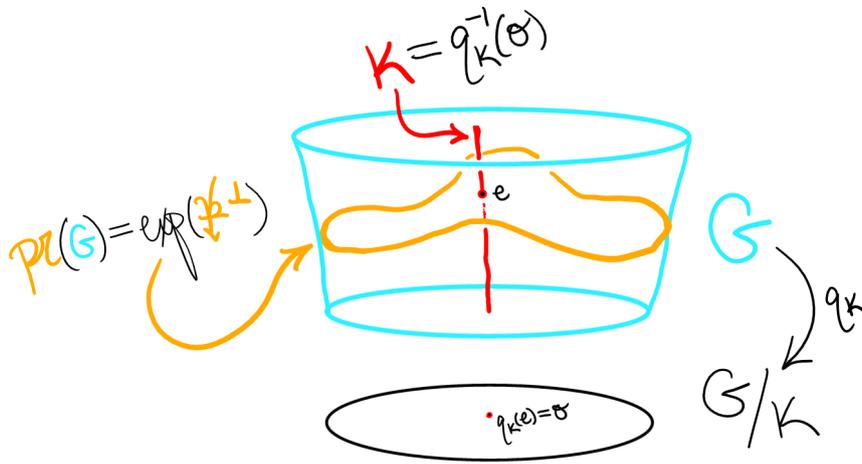


FIGURE 1. The projection  $\text{pr} : G \rightarrow \exp(\mathfrak{k}^\perp)$  gives a section to the natural quotient map  $q_K : G \rightarrow G/K$

The decompositions  $\mathfrak{g} = \mathfrak{k}^\perp \oplus \mathfrak{k}$  for the Lie algebra and  $G = \exp(\mathfrak{k}^\perp)K$  for the Lie group are both called the *Cartan decomposition* corresponding to  $K$ . At the level of Lie algebras, we can think of this as a decomposition into symmetric and skew-symmetric elements.

We've actually seen this before with hyperbolic geometry. In that case, our model  $(G, K)$  had  $G = \text{PO}(1, n)$  and  $K \simeq \text{O}(n)$ , and we

had a nice projection map that we used to give a diffeomorphism from  $\mathrm{PO}(1, n)/\mathrm{O}(n) = \mathbb{H}^n$  to  $\mathbb{R}^n$  after recognizing a subspace

$$\mathfrak{k}^\perp = \left\{ \begin{pmatrix} 0 & v^\top \\ v & 0 \end{pmatrix} : v \in \mathbb{R}^n \right\}$$

vaguely analogous to translations in the Euclidean case. It turns out that the symmetric space projection map  $\mathrm{pr} : \mathrm{PO}(1, n) \rightarrow \exp(\mathfrak{k}^\perp)$  for hyperbolic geometry is given by

$$\begin{pmatrix} a & \alpha \\ x & R \end{pmatrix} \mapsto \begin{pmatrix} a & x^\top \\ x & \mathbb{1} + \frac{1}{1+a}xx^\top \end{pmatrix},$$

and the image of this projection is uniquely determined by  $x \in \mathbb{R}^n$ . After identifying  $\mathrm{pr}(\begin{pmatrix} a & \alpha \\ x & R \end{pmatrix})$  with  $x$ , the induced map from  $\mathrm{PO}(1, n)/\mathrm{O}(n)$  to  $\mathbb{R}^n$  happens to coincide with that nice projection map that we used to identify  $\mathbb{H}^n \cong \mathbb{R}^n \cong \exp(\mathfrak{k}^\perp)$ .

For Cartan decompositions, our heuristic for the Killing form works exactly as expected: on the subalgebra  $\mathfrak{k}$  corresponding to the maximal compact subgroup,  $\mathfrak{h}$  is negative-definite, and on the subspace  $\mathfrak{k}^\perp$ , whose elements generate scaling transformations in the adjoint representation,  $\mathfrak{h}$  is positive-definite. This gives us an easy way of describing the pushforward projection  $\mathrm{pr}_* : \mathfrak{g} \rightarrow \mathfrak{k}^\perp$  at the identity: for  $X \in \mathfrak{g}$ , the projection  $\mathrm{pr}_*(X) \in \mathfrak{k}^\perp$  is the element  $X' \in X + \mathfrak{k}$  for which  $\mathfrak{h}(X', X')$  is maximal. Moreover,

$$\mathfrak{h}(X, Y) = \mathfrak{h}(\mathrm{pr}_*(X), \mathrm{pr}_*(Y)) + \mathfrak{h}(X - \mathrm{pr}_*(X), Y - \mathrm{pr}_*(Y)),$$

so we can genuinely decompose  $\mathfrak{h}(X, X)$  as the sum of the “scaling part” and the “compact part”, and for  $X \in \mathfrak{k}^\perp$ ,  $\mathfrak{h}(X, Y) = \mathfrak{h}(X, \mathrm{pr}_*(Y))$ .

Since  $\mathrm{pr}_*$  induces an isomorphism between  $\mathfrak{g}/\mathfrak{k}$  and  $\mathfrak{k}^\perp$ , we can identify the tangent bundle  $T(G/K) \cong G \times_K \mathfrak{g}/\mathfrak{k}$  with the homogeneous vector bundle  $G \times_K \mathfrak{k}^\perp$ . This isomorphism also gives us a canonical choice of Riemannian metric on  $G/K$ :  $\mathfrak{h}$  is positive-definite on  $\mathfrak{k}^\perp$ , so for  $X, Y \in T_{gK}(G/K)$ , we can define a Riemannian metric  $\mathrm{pr}^*\mathfrak{h}$  by

$$\mathrm{pr}^*\mathfrak{h}(X, Y) := \mathfrak{h}(\mathrm{pr}_*(L_{g^{-1}*} X), \mathrm{pr}_*(L_{g^{-1}*} Y)).$$

By construction, this is invariant under the canonical left-action of  $G$ , so it is a geometric object for the model.

Of course, for Riemannian manifolds, we get an associated notion of geodesic. As we did before with Euclidean geometry and hyperbolic geometry, though, we’ll define geodesics in terms of motion rather than the Riemannian metric. Specifically, we can think of  $\mathfrak{k}^\perp$  as being analogous to the subspace of translations in Euclidean geometry, and we define geodesics as (projections of) left-translates of one-parameter subgroups generated by elements of  $\mathfrak{k}^\perp$ .

**Definition 1.2.** A *geodesic* for the model geometry  $(G, K)$  is a curve  $\gamma : \mathbb{R} \rightarrow G/K$  of the form  $t \mapsto q_K(g \exp(tX))$  for some  $g \in G$  and  $X \in \mathfrak{k}^\perp$ .

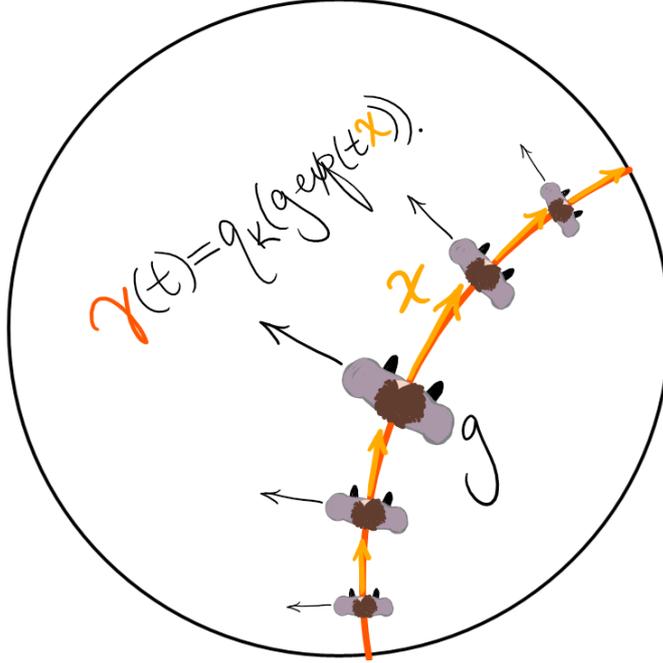


FIGURE 2. Geodesic motion corresponds to starting at some configuration in  $G$ , then moving with  $\omega_G$ -constant velocity in  $\mathfrak{k}^\perp$

As before, this corresponds to starting at some configuration  $g \in G$ , picking a velocity  $X \in \mathfrak{k}^\perp$ , and at each point in time moving with the velocity that the Maurer-Cartan form identifies with  $X$ , so that we move with “constant velocity”; by construction, every left-translate of a geodesic is again a geodesic, so geodesics are geometric for  $(G, K)$ . In this case, geodesics in our sense coincide with geodesics in the Riemannian sense.

This is, of course, not a very thorough introduction to the topic of Riemannian symmetric spaces of noncompact type. For such an introduction, we highly recommend [2].

## 2. ASYMPTOTIC BEHAVIOR OF GEODESICS

Another concept that makes sense for Riemannian manifolds is the *distance* between two points. Indeed, if we know the projection map  $\text{pr} : G \rightarrow \exp(\mathfrak{k}^\perp)$ , then distance is fairly straightforward to find in this case: for elements  $g_0, g_1 \in G$ , there is a unique  $X \in \mathfrak{k}^\perp$  such that  $\exp(X) = \text{pr}(g_0^{-1}g_1)$ , and the distance  $\text{dist}(q_K(g_0), q_K(g_1))$  from  $q_K(g_0)$  to  $q_K(g_1)$  is just  $\sqrt{\mathfrak{h}(X, X)}$ .

For us, the main use for this is to describe the asymptotic behavior of geodesics, since this will lead us to parabolic subgroups.

**Definition 2.1.** Suppose  $\gamma_1$  and  $\gamma_2$  are unit-speed geodesics in  $G/K$ . We say that  $\gamma_1$  and  $\gamma_2$  are *asymptotic* if and only if the distance  $\text{dist}(\gamma_1(t), \gamma_2(t))$  is bounded for  $t \geq 0$ .



FIGURE 3. Several asymptotic geodesics and their corresponding point at infinity

This defines an equivalence relation on geodesics, and an equivalence class of asymptotic geodesics is called a *point at infinity*.

**Definition 2.2.** An equivalence class of asymptotic geodesics is called a *point at infinity*. For a geodesic  $\gamma$ , we denote its corresponding point at infinity by  $\gamma(+\infty)$ .

Topologically, we can identify the space of all points at infinity with the unit sphere in  $\mathfrak{k}^\perp$ , since each  $Z \in \mathfrak{k}^\perp$  with  $\mathfrak{h}(Z, Z) = 1$  uniquely determines a unit-speed geodesic of the form  $t \mapsto q_K(\exp(tZ))$ , and every point at infinity corresponds to exactly one such geodesic.

### 3. PRELUDE TO PARABOLIC SUBGROUPS

In the trichotomy for elements of  $\mathfrak{sl}_2\mathbb{R}$  in terms of conic sections from last time, a considerably more well-known characterization of parabolic transformations is as transformations that fix a single point at infinity for the hyperbolic plane. With this in mind, it *almost* wouldn't be ridiculous to call the stabilizer subgroup of a point at infinity, or more generally a finite-index subgroup of such a stabilizer, a *parabolic subgroup*.

While this does give a mostly valid<sup>1</sup> definition for parabolic subgroups, it would be kind of annoying to use in practice. Imagine we found a closed subgroup of  $G$  and we wanted to check whether it was parabolic; without more information, we'd basically have to start checking geodesics to see whether their asymptotic behavior was preserved

<sup>1</sup>It should be noted that some larger subgroups, such as  $G$  itself, would be considered parabolic by most representation theorists, even though they don't fix a point at infinity. Our algebraic definition below accounts for this.

by our subgroup. We'd like a more direct definition, preferably one that comes from the structure of the Lie algebra.

In an attempt to ascertain such a definition, let's start with a point  $\gamma(+\infty)$  at infinity and try to find its stabilizer. As we mentioned above, we may assume that our geodesic  $\gamma$  is of the form  $t \mapsto q_K(\exp(tZ))$  for some  $Z \in \mathfrak{k}^\perp$ .

An element  $g \in G$  fixes  $\gamma(+\infty)$  if and only if  $g\gamma$  is asymptotic to  $\gamma$ . Thus, we want to find  $g \in G$  such that

$$g\gamma(t) = q_K(g \exp(tZ)) = q_K(\exp(tZ)(\exp(tZ)^{-1}g \exp(tZ)))$$

is a bounded distance away from  $\gamma(t) = q_K(\exp(tZ))$  for all  $t \geq 0$ , which amounts to showing that  $\text{pr}(\exp(tZ)^{-1}g \exp(tZ))$  is bounded. In particular, for  $g = \exp(X)$  for some  $X \in \mathfrak{g}$ , we have

$$\begin{aligned} \exp(tZ)^{-1}g \exp(tZ) &= \exp(tZ)^{-1} \exp(X) \exp(tZ) \\ &= \exp(\text{Ad}_{\exp(tZ)^{-1}}(X)), \end{aligned}$$

so we want to find  $X \in \mathfrak{g}$  such that  $\text{pr}_*(\text{Ad}_{\exp(tZ)^{-1}}(X))$  is bounded for  $t \geq 0$ .

Because  $\text{ad}_Z$  is diagonalizable over  $\mathbb{R}$ , we can decompose  $\mathfrak{g}$ , as a vector space, as  $\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$ , where  $\mathfrak{g}_-$  is the sum of all the negative eigenspaces for  $\text{ad}_Z$ ,  $\mathfrak{g}_0$  is the centralizer  $\mathfrak{z}_{\mathfrak{g}}(Z)$  of  $Z$ , and  $\mathfrak{p}_+$  is the sum of all the positive eigenspaces for  $\text{ad}_Z$ . Equivalently, we could define  $\mathfrak{g}_-$  as

$$\mathfrak{g}_- := \{X \in \mathfrak{g} : \text{Ad}_{\exp(tZ)}(X) \rightarrow 0 \text{ as } t \rightarrow +\infty\}$$

and  $\mathfrak{p}_+$  as

$$\mathfrak{p}_+ := \{X \in \mathfrak{g} : \text{Ad}_{\exp(tZ)^{-1}}(X) \rightarrow 0 \text{ as } t \rightarrow +\infty\}.$$

Because  $\text{Ad}_{\exp(tZ)}$  is an automorphism, both  $\mathfrak{g}_-$  and  $\mathfrak{p}_+$  are subalgebras, which homogeneous dynamicists would call the *contracting* and *expanding horospherical subalgebras* of  $Z$ , respectively. (See, for example, the excellent book [4].)

Crucially, note that  $\text{Ad}_g(X)$  has the same eigenvalues in the adjoint representation as  $X$ , so elements of the expanding and contracting horospherical subalgebras only have 0 as an eigenvalue under the adjoint representation, which means that elements of these subalgebras are always ad-nilpotent. In particular,  $\mathfrak{g}_-$  and  $\mathfrak{p}_+$  are always nilpotent subalgebras of  $\mathfrak{g}$ .

Writing  $X = X_- + X_0 + X_+$ , with  $X_- \in \mathfrak{g}_-$ ,  $X_0 \in \mathfrak{g}_0$ , and  $X_+ \in \mathfrak{p}_+$ , we see that

$$\text{pr}_*(\text{Ad}_{\exp(tZ)^{-1}}(X)) = \text{pr}_*(\text{Ad}_{\exp(tZ)^{-1}}(X_-) + X_0 + \text{Ad}_{\exp(tZ)^{-1}}(X_+))$$

is bounded for all  $t \geq 0$  if and only if  $X_- = 0$ . Thus, the Lie subalgebra of the stabilizer subgroup for  $\gamma(+\infty)$  is precisely  $\mathfrak{p} := \mathfrak{g}_0 + \mathfrak{p}_+$ .

Before moving on, it's well-worth trying to visualize this decomposition, since it will be very important from here onward.

Let us once again imagine ourselves as observers in the model group  $G$ , moving geodesically using right-translation by  $\exp(tZ)$ . At each configuration in  $G$ , we can use the Maurer-Cartan form  $\omega_G$  to decompose the tangent spaces according to the decomposition  $\mathfrak{g}_- + \mathfrak{g}_0 + \mathfrak{p}_+$  for  $\mathfrak{g}$ . Since  $\mathfrak{g}_-$ ,  $\mathfrak{g}_0$ , and  $\mathfrak{p}_+$  are subalgebras, the corresponding distributions are integrable, with integral submanifolds corresponding to the left-cosets of the connected subgroups generated by each subalgebra.

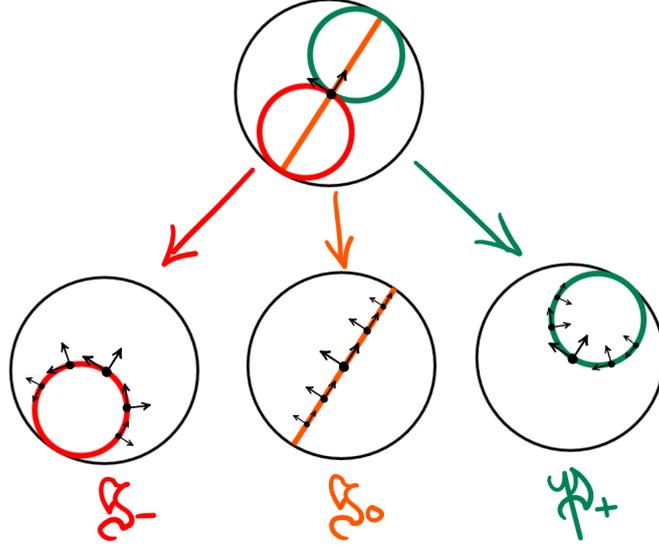


FIGURE 4. Using the Maurer-Cartan form  $\omega_G$ , we can decompose tangent spaces of  $G$  as sums of the integrable distributions  $\omega_G^{-1}(\mathfrak{g}_-)$ ,  $\omega_G^{-1}(\mathfrak{g}_0)$ , and  $\omega_G^{-1}(\mathfrak{p}_+)$

Let us denote by  $G_-$  and  $P_+$  the connected subgroups generated by  $\mathfrak{g}_-$  and  $\mathfrak{p}_+$ , respectively. Then, the integral submanifold for  $\omega_G^{-1}(\mathfrak{g}_-)$  through  $g \in G$  is precisely  $gG_-$ , and similarly,  $gP_+$  is the integral submanifold for  $\omega_G^{-1}(\mathfrak{p}_+)$  through  $g$ . As one might imagine from the term “horospherical subalgebra”, these left-cosets for  $G_-$  and  $P_+$  project to horospheres under the quotient map  $q_K$ .

Consider a starting configuration  $g \in G$  and an element  $p \in P_+$ , so that  $g$  and  $gp$  lie on the same integral submanifold for  $\omega_G^{-1}(\mathfrak{p}_+)$ . Then, moving by  $\exp(tZ)$  at both these points,  $g$  goes to  $g \exp(tZ)$  and  $gp$  goes to  $gp \exp(tZ) = g \exp(tZ)(\exp(tZ)^{-1}p \exp(tZ))$ . Essentially by definition of  $P_+$ ,  $\exp(tZ)^{-1}p \exp(tZ)$  will converge to the identity element as  $t \rightarrow +\infty$ , so  $g \exp(tZ)$  and  $gp \exp(tZ)$  get closer and closer together for larger and larger  $t$ . In other words, motion by  $\exp(tZ)$  contracts the leaves  $gP_+$ .

Similarly, motion by  $\exp(tZ)$  expands the leaves  $gG_-$  of the distribution  $\omega_G^{-1}(\mathfrak{g}_-)$ . We call the foliation from the distribution  $\omega_G^{-1}(\mathfrak{p}_+)$  the *stable foliation* for  $R_{\exp(tZ)}$  and the foliation from  $\omega_G^{-1}(\mathfrak{g}_-)$  the *unstable foliation* for  $R_{\exp(tZ)}$ .

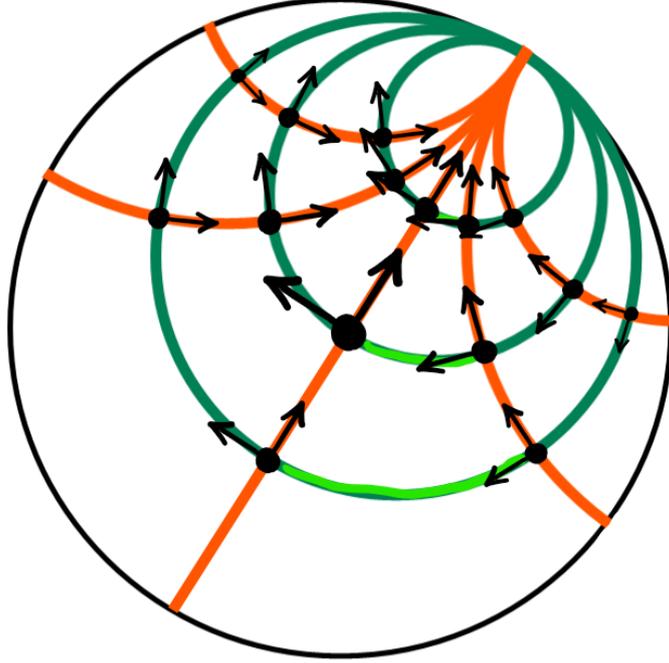


FIGURE 5. Moving by  $\exp(tZ)$  contracts the leaves of the stable foliation generated by  $\omega_G^{-1}(\mathfrak{p}_+)$

We should remark that homogeneous dynamicists are typically interested in the behavior of elements of  $G$  as *transformations*. Since we want to consider elements of  $G$  in terms of *motions* here—acting on the right so that we preserve left-invariance—the roles of the expanding and contracting horospherical subgroups are reversed: the left-cosets of the “expanding” horospherical subgroup  $P_+$  are contracted by moving by  $\exp(tZ)$ , and the left-cosets of the “contracting” horospherical subgroup  $G_-$  are expanded.

Since  $Z$  is centralized by  $\mathfrak{g}_0$ , motion by  $\exp(tZ)$  doesn’t affect the distribution  $\omega_G^{-1}(\mathfrak{g}_0)$ : for every  $X \in \mathfrak{g}_0$ ,  $R_{\exp(tZ)*}\omega_G^{-1}(X) = \omega_G^{-1}(X)$ . We call the foliation generated by  $\omega_G^{-1}(\mathfrak{g}_0)$  the *neutral foliation*. Since  $Z$  obviously centralizes itself, the leaf of this foliation through  $g \in G$  will contain the full geodesic trajectory  $g \exp(\mathbb{R}Z)$  of  $g$ . This allows us to imagine these leaves as “tubes” of asymptotic geodesic trajectories.

For  $X_1$  and  $X_2$  eigenvectors of  $\text{ad}_Z$  with respective eigenvalues  $\lambda_1$  and  $\lambda_2$ , we have

$$0 = \mathfrak{h}(\text{ad}_Z(X_1), X_2) + \mathfrak{h}(X_1, \text{ad}_Z(X_2)) = (\lambda_1 + \lambda_2)\mathfrak{h}(X_1, X_2),$$

so  $\mathfrak{h}(X_1, X_2) = 0$  unless  $\lambda_1 + \lambda_2 = 0$ . In particular, since  $\mathfrak{p}_+$  is the sum of the positive eigenspaces, this tells us that  $\mathfrak{p}_+$  is  $\mathfrak{h}$ -orthogonal to both itself and  $\mathfrak{g}_0$ , and similarly,  $\mathfrak{g}_-$  is  $\mathfrak{h}$ -orthogonal to both itself and  $\mathfrak{g}_0$ . Moreover, because  $\mathfrak{h}$  is nondegenerate, this also tells us that the eigenvalues of  $\text{ad}_Z$  must occur in pairs  $\pm\lambda$ , with the eigenspace for  $\lambda$

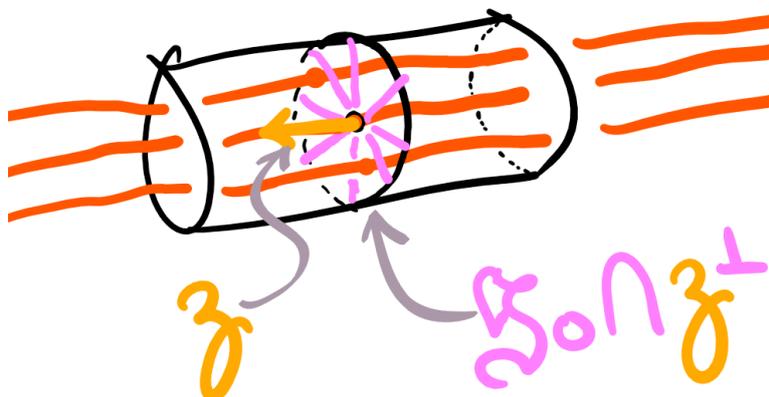


FIGURE 6. Each leaf of the neutral foliation generated by  $\omega_G^{-1}(\mathfrak{g}_0)$  consists of a tube of asymptotic geodesic trajectories of the form  $g \exp(\mathbb{R}Z)$

dual to the eigenspace for  $-\lambda$  with respect to  $\mathfrak{h}$ , and  $\mathfrak{h}$  must remain nondegenerate when restricted to the 0-eigenspace  $\mathfrak{g}_0$ .

To summarize the picture, we have a tube of asymptotic geodesic trajectories around each configuration  $g \in G$ , corresponding to the leaf of the neutral foliation through  $g$ , together with two left-cosets  $gP_+$  and  $gG_-$ , corresponding to leaves of the stable and unstable foliation respectively, that are  $\mathfrak{h}$ -orthogonal to the tube. Under the natural quotient map  $q_K$ , these left-cosets project to horospheres, with  $q_K(gP_+)$  a horosphere “centered” at the point at infinity given by following  $t \mapsto q_K(g \exp(tZ))$  as  $t \rightarrow +\infty$  and  $q_K(gG_-)$  a horosphere “centered” at the point at infinity given by following  $t \mapsto q_K(g \exp(tZ))$  as  $t \rightarrow -\infty$ . These horospheres are tangent at  $q_K(g)$ , and both are transverse to the image of the leaf of the neutral foliation.

In the semisimple case, we can get a lot of useful intuition for  $\mathfrak{h}$  from the duality between  $\mathfrak{p}_+$  and  $\mathfrak{g}_-$ . For each eigenspace  $\mathfrak{g}_\lambda$  of  $\text{ad}_Z$  with positive eigenvalue  $\lambda$ , there is another eigenspace  $\mathfrak{g}_{-\lambda}$  with negative eigenvalue  $-\lambda$ , and they pair together under the Killing form. In the picture above,  $\omega_G^{-1}(\mathfrak{g}_\lambda)$  is tangent to the stable foliation and  $\omega_G^{-1}(\mathfrak{g}_{-\lambda})$  is tangent to the unstable foliation, and they project to the same subspace of the tangent space under the natural quotient map  $q_K$ , so the pairing can sort of be seen from the canonical Riemannian metric being positive-definite.

For us, though, the crucial takeaway from this duality is what it tells us about the horospherical subalgebra  $\mathfrak{p}_+$ . We’ve already seen that  $\mathfrak{p}_+$  is  $\mathfrak{h}$ -orthogonal to both itself and  $\mathfrak{g}_0$ . Because each element  $Y \in \mathfrak{p}_+$  has an element  $X \in \mathfrak{g}_-$  for which  $\mathfrak{h}(X, Y) \neq 0$ , this then tells us that  $\mathfrak{p}_+^\perp$  is precisely the Lie subalgebra  $\mathfrak{p}$  of the stabilizer of the point at infinity. By nondegeneracy of  $\mathfrak{h}$ , we therefore have  $\mathfrak{p}^\perp = \mathfrak{p}_+$ .

We should think of the existence of these nilpotent, horospherical subalgebras  $\mathfrak{p}_+$  that are  $\mathfrak{h}$ -orthogonal to all of  $\mathfrak{p}$  as the defining characteristic of parabolic subalgebras. Indeed, their existence is precisely the property that we will use to define parabolicity.<sup>2</sup>

**Definition 3.1.** A subalgebra  $\mathfrak{p} \leq \mathfrak{g}$  is *parabolic* if and only if  $\mathfrak{p}^\perp$  is a nilpotent subalgebra. A *parabolic subgroup*  $P \leq G$ , then, is a closed subgroup whose Lie subalgebra  $\mathfrak{p}$  is parabolic.

Next time, we will take this definition and attempt to build some useful tools for working with parabolic subgroups. Additionally, we will construct a fixed point at infinity for each (proper) parabolic subgroup.

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<sup>2</sup>Since this is a somewhat less well-known definition of parabolicity, I should note that I essentially learned this while perusing [1], which attributes it to Grothendieck and Burstall in papers that I was unable to find.